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On the equivalence of heat kernel estimates and logarithmic Sobolev inequalities for the Hodge Laplacian ☆

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Abstract

In this paper we consider the Hodge Laplacian on differential k-forms over smooth open manifolds M^N, not necessarily compact. We find sufficient conditions under which the existence of a family of logarithmic Sobolev inequalities for the Hodge Laplacian is equivalent to the ultracontractivity of its heat operator.

We will also show how to obtain a logarithmic Sobolev inequality for the Hodge Laplacian when there exists one for the Laplacian on functions. In the particular case of Ricci curvature bounded below, we use the Gaussian type bound for the heat kernel of the Laplacian on functions in order to obtain a similar Gaussian type bound for the heat kernel of the Hodge Laplacian. This is done via logarithmic Sobolev inequalities and under the additional assumption that the volume of balls of radius one is uniformly bounded below.

Keywords: Hodge Laplacian; Heat operator; Bochner technique; Logarithmic Sobolev inequalities; Ultracontractivity

1. Introduction

The usefulness of logarithmic Sobolev inequalities was first demonstrated by Gross, who showed that they are equivalent to the hypercontractivity of the heat operator on $\mathbb{R}^N$ with the Gauss measure [6]. Since then, such equivalences have been proved for a larger class of operators and spaces. In the particular case of the Bochner Laplacian on k-forms over manifolds with a probability metric, Qian has been able to show that the existence of a family of logarithmic

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Sobolev inequalities for the Bochner Laplacian is equivalent to the hypercontractivity of its heat operator [7]. The equivalence of logarithmic Sobolev inequalities to the ultracontractivity of the heat kernel of an operator is usually harder to prove. Davies proves such equivalences for the Dirichlet Laplacian on $\mathbb{R}^N$ [4].

The main result of this paper is the extension of such equivalences to the case of the Hodge Laplacian acting on differential k-forms. We find sufficient conditions under which the existence of a family of logarithmic Sobolev inequalities for Hodge Laplacian implies an ultracontractive bound for its heat operator and vice versa. This will be done on smooth open manifolds with Weitzenböck tensor bounded below. The key step in the proof consists of expressing the Hodge Laplacian using the Bochner Weitzenböck formula. In this way, we split the operator into a self-adjoint operator with a contractive heat kernel, namely the Bochner Laplacian, plus the Weitzenböck tensor which we treat as a potential term that can be bounded below.

In Section 3 we prove some intermediate results. We show that the heat operator of the Hodge Laplacian is bounded from L^2 to L^∞ at a specific moment in time whenever the Hodge Laplacian satisfies an appropriate family of logarithmic Sobolev inequalities (Theorem 3.1). We also prove a similar bound for a special perturbation of this heat operator (Corollary 3.2).

In Section 4 we use stronger curvature assumptions in order to extend these bounds. The manifolds considered have the Weitzenböck tensor and Ricci curvature bounded below by a non-positive constant, and the volume of balls of radius one bounded below by a uniform constant. We prove that on these manifolds the Hodge Laplacian on forms satisfies a family of logarithmic Sobolev inequalities with coefficients that allow us to conclude that its heat operator is ultracontractive (Theorem 4.2). Furthermore, the bound of the perturbed heat operator from Corollary 3.2 generalizes and gives us a Gaussian type bound for the heat operator of the Hodge Laplacian (Theorem 4.3).

One of the main elements of these results is proving that a logarithmic Sobolev inequality for the Laplacian on functions implies a similar inequality for the Hodge Laplacian on forms, whenever the Weitzenböck tensor is bounded below (Lemma 4.1). This essentially allows us to demonstrate that a Gaussian type heat kernel bound for the Laplacian on functions implies the same type of bound for the heat kernel of the Hodge Laplacian on forms, on the class of manifolds we consider in Section 4.

Furthermore, we demonstrate how to obtain a family of logarithmic Sobolev inequalities for the Hodge Laplacian whenever its heat operator is ultracontractive (Theorem 4.1). This is the second part of the equivalence between the existence of logarithmic Sobolev inequalities and ultracontractive heat operator bounds on forms.

The author developed the results of this paper when looking for a Gaussian type of heat kernel bound for the Hodge Laplacian in order to prove the L^p independence of its spectrum. A similar bound, also known as Kato's inequality, is due to Rosenberg [8]. The author was initially unaware of Rosenberg's work and developed this independent method for obtaining Gaussian bounds for the Hodge Laplacian. Rosenberg's result allows for a slightly better Gaussian bound and was proved via the Feynman–Kac formula. Nonetheless, the bound in this paper is sufficient for proving the L^p independence of the spectrum under the further assumption on the volume of balls of radius one [2]. As we mention at the end of Section 4, the merit of this technique is that it has the potential of being generalized to a larger class of self-adjoint operators. As a result, it can be used to extend the L^p independence results to these operators.

2. Preliminaries

Denote by $\Lambda^k(M^N)$, the space of differential k-forms on the Riemannian manifold (M^N, g). The metric g induces a pointwise inner product on k-forms which we denote $(\cdot,\cdot)$. The L^2 inner product will be denoted by $\langle\cdot,\cdot\rangle = \int_M (\cdot,\cdot)$.

Definition 2.1. A self-adjoint operator H on k-forms with quadratic form Q satisfies a logarithmic Sobolev inequality if there exist constants ε and β such that

$$\int_M |\phi|^2 \log|\phi| \leqslant \varepsilon Q(\phi) + \beta\|\phi\|_2^2 + \|\phi\|_2^2 \log\|\phi\|_2 \tag{S2}$$

for ϕ in $\mathrm{Dom}(Q) \cap L^1 \cap L^\infty$. $|\phi|$ denotes the pointwise Riemannian norm of the form ϕ and $\mathrm{Dom}(Q) = \mathrm{Dom}(H^{1/2})$.

The operator satisfies a p-logarithmic Sobolev inequality for $2 < p < \infty$ if there exist constants a and b such that

$$\int_M |\phi|^p \log|\phi| \leqslant a\big\langle|\phi|^{p-2}\phi, H\phi\big\rangle + b\|\phi\|_p^p + \|\phi\|_p^p \log\|\phi\|_p. \tag{Sp}$$

In this paper, H will be the Hodge Laplacian Δ^k on differential k-forms. The results from Sections 3 and 4 can be easily generalized to operators of the type $\Delta^k + T^k$ where $T^k : \Lambda^k(M) \to \Lambda^k(M)$ is a tensor acting pointwise on differential forms, and which we can bound below in the following sense: we consider the action of T^k on $\Lambda^k(M) \times \Lambda^k(M)$ by which T^k maps two k-forms ω and η to $(T^k\omega, \eta)$. We say that the tensor T^k is bounded below if there exists a uniform constant C such that $(T^k\omega, \omega)_y \geqslant C|\omega|_y^2$ at all points $y \in M$.

On smooth differential k-forms with compact support, the Hodge Laplacian is defined as $\Delta^k = \delta d + d\delta$ where d is exterior differentiation and δ is its adjoint as an operator on forms with respect to the inner product $\langle\cdot,\cdot\rangle$. We will usually just write Δ when the context is clear and denote as Δ^0 the Laplacian on functions.

For a locally defined orthonormal frame field $\{V_i\}_i$ with dual coframe $\{\omega^j\}_j$ defined by $\omega^j(V_i) = \delta_i^j$, these operators are given by the formulas:

$$d = \sum_j \omega^j \wedge D_{V_j}, \qquad \delta = -\sum_i i(V_i) D_{V_i}$$

where D_X is the Levi-Civita connection and $i(V_i)$ denotes the interior product [12].

If ω, η are C^∞ k-forms in M with compact support, then $\langle\Delta^k\omega, \eta\rangle = \langle d\omega, d\eta\rangle + \langle\delta\omega, \delta\eta\rangle$.

The Hodge Laplacian can also be written via the Weitzenböck formula:

$$\Delta = -\sum_i D^2_{V_i V_i} - \sum_{i,j} \omega^i \wedge i(V_j) R_{V_i V_j}$$

where $D^2_{XY} = D_X D_Y - D_{D_X Y}$ is the second order covariant differential and $R_{XY} = D_X D_Y - D_Y D_X - D_{[X,Y]}$ is the curvature tensor.

Definition 2.2. The operator $\mathcal{W}^k = -\sum_{i,j} \omega^i \wedge i(V_j) R_{V_i V_j}$ is the Weitzenböck tensor, acting on k-forms as a derivation.

A simple calculation shows that for one-forms $(\mathcal{W}^1\eta, \eta)_x = \mathrm{Ric}_x(\eta^*, \eta^*)$ where η^* is the vector dual of the one-form η [12].

We state a few results of Gallott and Meyer [5] in order to provide some insight for this tensor. Define the curvature operator $\rho_{x_0} : T_{x_0}M \times T_{x_0}M \to \mathbb{R}$ by $\rho(X, Y)_{x_0} = (R_{XY}Y, X)_{x_0}$, for vectors X, Y in $T_{x_0}M$. Note that a lower bound on sectional curvature does not imply a lower bound for the curvature operator. Now let X^*, Y^*, be the dual covectors to X, Y respectively. If $\rho(X, Y)_{x_0} \geqslant \lambda|X^* \wedge Y^*|^2$ for some $\lambda \in \mathbb{R}$ and for all vectors X, Y at a point x_0, then $(\mathcal{W}^k\eta, \eta)_{x_0} \geqslant k(N-k)\lambda|\eta|^2$ on each k-form η. In other words, the Weitzenböck tensor is bounded below whenever the curvature operator is. Although the lower bound on the curvature operator is more common in the literature, we keep the weaker assumption on the lower bound of the Weitzenböck tensor.

We let $D^2 = \sum_i D^2_{V_i V_i}$ and in order to avoid a repeated use of summands, we write the pointwise inner product $\sum_i (D_{V_i}\omega, D_{V_i}\eta) = (\bar{D}\omega, \bar{D}\eta)$.

Note that in the case of functions $(\bar{D}f, \bar{D}g) = (df, dg)$.

It follows that $\int_M (-D^2\omega, \eta) = \int_M (\bar{D}\omega, \bar{D}\eta)$ for all C^∞ forms ω, η when at least one of the two has compact support [7].

Definition 2.3. The operator $\mathcal{L}^k = -D^2$ is called the Bochner Laplacian. It is a non-negative symmetric operator on smooth forms with compact support.

We consider the L^2 closure of $\mathcal{L}^k$ on smooth k-forms with compact support, denoted as $\mathcal{L}^k_2$, and let $D(\mathcal{L}^k_2)$ be the domain of this extension. Strichartz proves that the L^2 closure of the Hodge Laplacian Δ^k is self-adjoint on a complete manifold [11]. His argument easily generalizes to the Bochner Laplacian. As a result, $\mathcal{L}^k_2$ is a closed and self-adjoint operator on $D(\mathcal{L}^k_2)$ and $D(\mathcal{L}^k_2) \subset \mathrm{Dom}(Q_{\mathcal{L}^k_2}) = \mathrm{Dom}((\mathcal{L}^k_2)^{1/2})$ [4]. For simplicity we will usually denote the operator $\mathcal{L}^k_2$ by $\mathcal{L}$.

We now rewrite the Hodge Laplacian as

$$\Delta^k = \mathcal{L}^k + \mathcal{W}^k$$

in the form of a symmetric operator plus a potential-type term. This is the standard format that was used by Rosenberg in his proof of Kato's inequality [8].

Whenever $\mathcal{W}^k$ is bounded below by a constant $-K_1$, we show that the L^2 closure of $\Delta^k = \mathcal{L}^k + \mathcal{W}^k$ on $C^\infty_c(\Lambda^k)$ is also self-adjoint. Using the same argument as in [11] it follows that the L^2 closure of $A = \mathcal{L}^k + \mathcal{W}^k + K_1\,\mathrm{Id}$ on $C^\infty_c(\Lambda^k)$ is self-adjoint. $K_1\,\mathrm{Id}$ is a bounded self-adjoint operator on L^2. It follows that the L^2 closure of Δ^k on smooth k-forms with compact support is a closed and self-adjoint operator. We denote this closure by Δ^k_2. Furthermore, the domain of Δ^k_2, $D(\Delta^k_2)$, satisfies the property $D(\Delta^k_2) \subset \mathrm{Dom}(Q_{\Delta^k_2}) = \mathrm{Dom}((\Delta^k_2)^{1/2})$ [4].

Inequality (17) from Section 4 demonstrates that $D(\Delta^k_2) \subset D(\mathcal{L}^k_2)$ and $\mathrm{Dom}(Q_{\Delta^k_2}) \subset \mathrm{Dom}(Q_{\mathcal{L}^k_2})$.

For simplicity in notation we will sometimes talk about the action of Δ^k on $L^2(\Lambda^k)$ referring to the L^2 closure of Δ^k acting on $D(\Delta^k_2)$. Furthermore, we generalize Δ^k to the space $L^p(\Lambda^k)$ by considering its action on $L^2 \cap L^p(\Lambda^k)$ and seeing whether it can be extended to $L^p(\Lambda^k)$.

This poses no problems when $1 \leqslant p \leqslant \infty$ under the additional assumption that $\Delta^k_\infty = (\Delta^k_1)^*$. We denote as Δ^k_p the operator Δ^k acting on $L^p(\Lambda^k)$.

For an operator H we denote by $\|H\|_{\beta,\alpha}$ its norm from L^α to L^β. We say that its heat operator e^{-tH} is ultracontractive if it is bounded from L^2 to L^∞ for all $t > 0$. If $\|e^{-tH}\|_{p,p} \leqslant 1$ for all $t > 0$ then we say that e^{-tH} is a contraction on L^p.

The heat operator $e^{-t\Delta^k}$ of the Hodge Laplacian is a linear operator from $\Lambda^k(T_xM)$ to $\Lambda^k(T_yM)$. It is well defined on $D(\Delta^k)$ where Δ^k is self-adjoint. When the kernel of the heat operator exists, we will denote it by $\vec{P}_k(t,x,y)$, and we denote as $P(t,x,y)$ the heat kernel for the Laplacian on functions.

We make the following remark found in [9]: Let $I = (i_1, \dots, i_k)$ be a multi-index and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Then we may express $\vec{P}_k(t,x,y)$ in matrix form on coordinate patches U, V with $x \in U$ and $y \in V$ as

$$\vec{P}_k(t,x,y) = \sum_{I,J} P_{IJ}(x,y)\, dx^I \otimes\, dy^J.$$

The heat operator acts on a smooth k-form η with support on U via the following formula

$$e^{-t\Delta^k}(\eta)(y) = \sum_J \int_U \sum_I P_{IJ}(x,y)\big(dx^I, \eta(x)\big)\, dx \cdot dy^J.$$

Definition 2.4. We say that the heat kernel of the Hodge Laplacian has a pointwise bound $A(x,y)$, if $|P_{IJ}(x,y)| \leqslant A(x,y)$ for all I, J with $\{\omega^I(.)\}_I$ an orthonormal basis of k-forms at each point.

3. Finding upper bounds for the heat operator via logarithmic Sobolev inequalities

From now on we assume that the underlying manifold M^N is open with $\mathcal{W}^k \geqslant -K_1$ for some non-negative constant K_1.

In this section we prove an intermediate upper bound for the heat operator when the Hodge Laplacian satisfies a family of logarithmic Sobolev inequalities with appropriate constants. More specifically, we find a specific time t_0 such that $e^{-t_0\Delta^k}$ is bounded from L^2 to L^∞. We also demonstrate that a certain perturbation of $e^{-t\Delta^k}$ has a similar bound at some moment t_0.

Under stronger curvature conditions, the bound for $e^{-t_0\Delta^k}$ and its perturbation can be generalized to all $t_0 > 0$. The bound of the perturbed heat operator yields a Gaussian type bound for $\vec{P}_t$. We compute these bounds in Section 4 where we also impose the stronger conditions on the geometry of the manifold.

As we will see in Theorem 3.1, the intermediate bound for $e^{-t_0\Delta^k}$ follows from the existence of a family of appropriate (Sp) inequalities for Δ^k for all $2 < p < \infty$. The moment t_0 will depend on the coefficients of these (Sp) inequalities. Lemma 3.2 proves that it is in fact sufficient to have a family logarithmic Sobolev inequalities for $p = 2$. We include the proof of this Lemma before Theorem 3.1, as the constants that we obtain affect the bounds of $e^{-t_0\Delta^k}$ and we will need them in the calculations of Section 4.

We begin by proving a simple upper bound for $e^{-t\Delta^k_p}$ on these manifolds. This result is necessary for generalizing our domain (see Lemma 3.2). It also becomes crucial for obtaining the Gaussian bounds in Section 4.

Lemma 3.1. *Whenever* $\mathcal{W}^k \geqslant -K_1$, $e^{-t\Delta^k}$ *is bounded on* L^p *for all* $1 \leqslant p \leqslant \infty$ *and for each* $t > 0$. *Furthermore,* $\|e^{-tH}\|_{p,p} \leqslant e^{tK_1}$.
In particular, if $K_1 = 0$ *the heat operator of* Δ^k *is a contraction on* L^p *for all* $1 \leqslant p \leqslant \infty$.

Proof. Recall that $\Delta^k_\infty = (\Delta^k_1)^*$ and note that the heat operator generalizes to the L^p space in the same sense as Δ^k.
It suffices to check that the heat operator is bounded on L^∞. By duality, the heat operator will be bounded on L^1 and will have the same norm. Applying the Riesz–Thorin Interpolation Theorem we can conclude that it is also bounded on the remaining L^p spaces for $1 \leqslant p \leqslant \infty$ with the same norm. The Riesz–Thorin Interpolation Theorem is a special case of the Stein Interpolation Theorem, and a proof that it holds for operators acting on forms can be found in [1].
The heat operator is defined as

$$\exp(-t\Delta)\omega = \lim_{n\to\infty}\left(1 + \frac{t}{n}\Delta\right)^{-n}\omega.$$

Let ω be a k-form on M and normalize its norm such that $\|\omega\|_\infty \leqslant 1$. Set $s = \frac{t}{n}$, and take n large enough so that $sK_1 < 1$. As a result, $\eta = (1 + s\Delta)^{-1}\omega$ is well defined. It will suffice to show that the norm of η remains bounded.
From η we define

$$\bar{\eta} := \begin{cases} \eta & \text{if } |\eta| < \frac{1}{1-sK_1}, \\ \frac{\eta}{|\eta|}\frac{1}{1-sK_1} & \text{otherwise} \end{cases}$$

and let $A = \{x\colon\ |\eta| \geqslant \frac{1}{1-sK_1}\}$. Pointwise on A

$$(\omega, \eta - \bar{\eta}) \leqslant |\omega||\eta - \bar{\eta}| \leqslant |\eta - \bar{\eta}| \leqslant |\eta| - \frac{1}{1 - sK_1}. \tag{1}$$

We also obtain

$$\begin{aligned} \int_A (\omega, \eta - \bar{\eta})\,dv &= \int_A \big((1 + s\Delta)\eta, \eta - \bar{\eta}\big)\,dv \\ &= \int_A (\eta, \eta - \bar{\eta})\,dv - s\int_A \sum_i (D_{V_i}D_{V_i}\eta, \eta - \bar{\eta})\,dv + s\int_A (\mathcal{W}\eta, \eta - \bar{\eta}) \\ &\geqslant \int_A \left[|\eta|^2 - |\eta|\frac{1}{1 - sK_1}\right]dv - s\int_A \sum_i D_{V_i}(D_{V_i}\eta, \eta - \bar{\eta})\,dv \\ &\quad + s\int_A \sum_i \big(D_{V_i}\eta, D_{V_i}(\eta - \bar{\eta})\big)\,dv - sK_1\int_A |\eta|^2\left[1 - \frac{1}{|\eta|(1 - sK_1)}\right]. \end{aligned} \tag{2}$$

The second integral in the last inequality of (2) is in fact zero. To see that, we let $\{\omega^j\}_j$ be the dual basis to the basis of normal coordinates $\{V_i\}_i$ at a point x_0 and we define $\beta = \sum_i (D_{V_i}\eta, \eta - \bar{\eta})\omega^i$. At x_0, $\delta\beta = -\sum_i D_{V_i}(D_{V_i}\eta, \eta - \bar{\eta})$. It follows that

$$\int_A \delta\beta\, dv = \int_A d(*\beta) = \int_{\partial A} (*\beta)\, d\sigma.$$

As $\eta - \bar{\eta} = 0$ on ∂A, β is zero on ∂A, and the second integral of (2) is zero as claimed.

Furthermore, the third integral of (2) is non-negative

$$\begin{aligned} s\int_A \sum_i \big(D_{V_i}\eta, D_{V_i}(\eta - \bar{\eta})\big) &= s\int_A \left[(\bar{D}\eta, \bar{D}\eta) - \left(\bar{D}\eta, \frac{\bar{D}\eta}{|\eta|(1 - sK_1)}\right) + \frac{(\sum_i (D_{V_i}\eta, \eta))^2}{|\eta|^3(1 - sK_1)}\right] \\ &\geqslant s\int_A (\bar{D}\eta, \bar{D}\eta)\left(1 - \frac{1}{|\eta|(1 - sK_1)}\right) \geqslant 0. \end{aligned}$$

Using the value of the norm of η on A for the remaining two terms of (2)

$$\begin{aligned} \int_A (\omega, \eta - \bar{\eta})\, dv &\geqslant \int_A (1 - sK_1)|\eta|\left[|\eta| - \frac{1}{1 - sK_1}\right] dv \\ &\geqslant \int_A \left[|\eta| - \frac{1}{1 - sK_1}\right] dv. \end{aligned}$$

However, by Eq. (1)

$$\int_A (\omega, \eta - \bar{\eta})\, dv \leqslant \int_A \left[|\eta| - \frac{1}{1 - sK_1}\right] dv.$$

The two inequalities imply that $|\eta| = \frac{1}{1 - sK_1}$ on A.
In other words, $\|(1 + s\Delta)^{-1}\omega\|_\infty \leqslant \frac{1}{1 - sK_1}$ for s large enough.
From the definition of the heat operator and for n large enough

$$\big\|e^{-t\Delta}\omega\big\|_\infty = \lim_{n\to\infty} \left\|\left(1 + \frac{t}{n}\Delta\right)^{-n}\omega\right\|_\infty \leqslant \lim_{n\to\infty}\left(1 - \frac{t}{nK_1}\right)^{-n} = e^{tK_1}. \qquad \square$$

The following lemma shows how we may obtain the inequality (Sp) when (S2) holds for the Hodge Laplacian. The lower bound of the Weitzenböck tensor is key in the proof. The lemma also gives us an appropriate domain for our results.

Lemma 3.2. *Suppose that the logarithmic Sobolev inequality* (S2) *holds for the Hodge Laplacian on all differential k-forms $\omega \in \mathrm{Dom}(Q_{\Delta^k}) \cap L^1 \cap L^\infty$ and with constants ε and β'.*
Then (Sp) *holds for any $2 < p < \infty$ with $a = \varepsilon \frac{p}{2(p-1)}$ and $b = \beta' \frac{2}{p} + \sup\{\varepsilon \frac{pK_1}{2(p-1)}, 0\}$ for all differential k-forms $\omega \in \bigcup_{t>0} e^{-t\Delta^k}(L^1 \cap L^\infty)$, whenever $\mathcal{W}^k \geqslant -K_1$.*

Proof. Firstly, let us remark that $\bigcup_{t>0} e^{-t\Delta^k}(L^1 \cap L^\infty)$ is contained in $\mathrm{Dom}(Q_{\Delta^k}) \cap L^1 \cap L^\infty$. It is also dense in L^2, as $e^{-t\Delta}\omega$ converges to ω as $t \to 0$ for all C_c^∞ forms.
Some of the arguments in the proof are similar to those due to Qian [7] for the Bochner Laplacian on manifolds with a probability metric and Davies [4] for symmetric operators on functions, but they are much more general.
For $\omega \in C_c^\infty(\Lambda^k)$, after substituting $|\omega|^{\frac{p}{2}-1}\omega$ into the left-hand side of (S2), we obtain

$$\frac{p}{2}\int_M |\omega|^p \log|\omega| \leqslant \varepsilon Q_\Delta\big(|\omega|^{\frac{p}{2}-1}\omega\big) + \beta'\|\omega\|_p^p + \frac{p}{2}\|\omega\|_p^p \log\|\omega\|_p. \tag{3}$$

We will first show

$$\int_M \big|\bar{D}\big(|\omega|^{\frac{p}{2}-1}\omega\big)\big|^2 \leqslant \frac{p^2}{4(p-1)}\big\langle|\omega|^{p-2}\omega, \mathcal{L}\omega\big\rangle. \tag{4}$$

We compute that

$$\big|d\big(|\omega|^{\frac{p}{2}-1}\big)\big|^2|\omega|^2 = \frac{(p-2)^2}{4}|\omega|^{p-2}\big|d|\omega|\big|^2$$

whereas

$$\big|d\big(|\omega|^{\frac{p}{2}}\big)\big|^2 = \frac{p^2}{4}|\omega|^{p-2}\big|d|\omega|\big|^2.$$

Combining the two equalities, we obtain

$$\big|d\big(|\omega|^{\frac{p}{2}-1}\big)\big|^2|\omega|^2 = \frac{(p-2)^2}{p^2}\big|d\big(|\omega|^{\frac{p}{2}}\big)\big|^2.$$

Furthermore,

$$\begin{aligned}\big\langle\big(\bar{D}|\omega|^{p-2}\big)\omega, \bar{D}\omega\big\rangle &= \big\langle\big(\bar{D}|\omega|^{2(\frac{p}{2}-1)}\big)\omega, \bar{D}\omega\big\rangle \\ &= 2\big\langle|\omega|^{\frac{p}{2}-1}\bar{D}\big(|\omega|^{\frac{p}{2}-1}\big)\omega, \bar{D}\omega\big\rangle.\end{aligned}$$

With the above equalities, we find the following identity for C_c^∞ k-forms

$$\begin{aligned}\int_M \big|\bar{D}\big(|\omega|^{\frac{p}{2}-1}\omega\big)\big|^2 &= \int \big|\big(\bar{D}|\omega|^{\frac{p}{2}-1}\big)\omega + |\omega|^{\frac{p}{2}-1}\bar{D}\omega\big|^2\,dv \\ &= \int_M \big[\big|d\big(|\omega|^{\frac{p}{2}-1}\big)\big|^2|\omega|^2 + 2\big(|\omega|^{\frac{p}{2}-1}\bar{D}\big(|\omega|^{\frac{p}{2}-1}\big)\omega, \bar{D}\omega\big)\end{aligned}$$

$$+ |\omega|^{p-2}|\bar{D}\omega|^2\big]\,dv$$
$$= \frac{(p-2)^2}{p^2}\int_M \big|d\big(|\omega|^{\frac{p}{2}}\big)\big|^2\,dv + \big\langle |\omega|^{p-2}\omega, \mathcal{L}\omega\big\rangle. \tag{5}$$

For $p > 1$ define

$$U(p) = \frac{p^2}{p-1}\left[|\bar{D}\omega|^2 - \frac{(p-2)^2}{p^2}\big|d|\omega|\big|^2\right].$$

Differentiating at each ω with respect to p we get

$$U'(p) = \frac{p(p-2)}{(p-1)^2}\big[|\bar{D}\omega|^2 - \big|d|\omega|\big|^2\big].$$

Thus $U(p)$ is minimal when $p = 2$ or when $|\bar{D}\omega|^2 = |d|\omega||^2$ (in which case $U(p)$ is constant equal to $4|\bar{D}\omega|^2$). As a result, its minimum value is $4|\bar{D}\omega|^2$.

For the form $|\omega|^{\frac{p}{2}-1}\omega$, with $p > 2$, this result implies

$$\frac{p^2}{p-1}\left[\int_M \big|\bar{D}\big(|\omega|^{\frac{p}{2}-1}\omega\big)\big|^2 - \frac{(p-2)^2}{p^2}\int \big|d\big(|\omega|^{\frac{p}{2}}\big)\big|^2\,dv\right] \geqslant 4\int_M \big|\bar{D}\big(|\omega|^{\frac{p}{2}-1}\omega\big)\big|^2.$$

Hence

$$\frac{p^2}{p-1}\big\langle |\omega|^{p-2}\omega, \mathcal{L}\omega\big\rangle \geqslant 4\int_M \big|\bar{D}\big(|\omega|^{\frac{p}{2}-1}\omega\big)\big|^2$$

by Eq. (5). We have thus proved (4) on $C_c^\infty(\Lambda^k)$.

We remark that inequality (4) implies that

$$Q_\Delta\big(|\omega|^{\frac{p}{2}-1}\omega\big) \leqslant \frac{p^2}{4(p-1)}\big\langle |\omega|^{p-2}\omega, \mathcal{L}\omega\big\rangle + \big\langle |\omega|^{\frac{p}{2}-1}\omega, \mathcal{W}\big(|\omega|^{\frac{p}{2}-1}\omega\big)\big\rangle$$
$$= \frac{p^2}{4(p-1)}\big\langle |\omega|^{p-2}\omega, \Delta\omega\big\rangle + \left(1 - \frac{p^2}{4(p-1)}\right)\big\langle |\omega|^{p-2}\omega, \mathcal{W}\omega\big\rangle. \tag{6}$$

The definition of the heat operator however, transforms the above inequality into

$$\lim_{t\to 0^+}\frac{1}{t}\big\langle |\omega|^{\frac{p}{2}-1}\omega, \big(1 - e^{-t\Delta}\big)|\omega|^{\frac{p}{2}-1}\omega\big\rangle$$
$$\leqslant 4\frac{p^2}{4(p-1)}\lim_{t\to 0^+}\frac{1}{t}\big\langle |\omega|^{p-2}\omega, \big(1 - e^{-t\Delta}\big)\omega\big\rangle + \left(1 - \frac{p^2}{4(p-1)}\right)\big\langle |\omega|^{p-2}\omega, \mathcal{W}\omega\big\rangle.$$

The result clearly holds on $C_c^\infty(\Lambda^k)$ and this expression, together with the boundedness of the heat operator that we have proved in Lemma 3.1, demonstrate that we can generalize the inequality to forms $\omega \in \bigcup_{t>0} e^{-t\Delta^k}(L^1 \cap L^\infty)$, as this set is contained in $\mathrm{Dom}(Q_{\Delta^k}) \cap L^1 \cap L^\infty$. Thus we have obtained (6) on the desired domain.

From inequalities (3) and (6) we can now get the p-logarithmic Sobolev inequality

$$\int_M |\omega|^p \log|\omega| \leqslant \varepsilon \frac{p}{2(p-1)} \langle |\omega|^{p-2}\omega, \Delta\omega \rangle$$
$$+ \frac{2\varepsilon}{p}\left(1 - \frac{p^2}{4(p-1)}\right)\langle |\omega|^{p-2}\omega, \mathcal{W}\omega\rangle + \beta' \frac{2}{p}\|\omega\|_p^p + \|\omega\|_p^p \log\|\omega\|_p.$$

The result with the claimed constants follows after using the lower bound on $\mathcal{W}^k$ and that $(p-2)^2 \leqslant p^2$.

It is worth noting that Qian [7] proved the equivalence between (S2) and (Sp) in the case of forms in $C_c^\infty(\Lambda^k)$ for the Bochner Laplacian. $\square$

We are now ready to state and prove the intermediate bound for $e^{-t\Delta^k}$.

Theorem 3.1. *Suppose that for all $\omega \in \bigcup_{t>0} e^{-t\Delta^k}(L^1 \cap L^\infty)$ and $2 < p < \infty$ the following* (Sp) *inequality holds*

$$\int |\omega|^p \log|\omega| \leqslant \bar{\varepsilon}(p) \int_M |\omega|^{p-2}\big(\Delta^k\omega, \omega\big) + \Gamma(p)\|\omega\|_p^p + \|\omega\|_p^p \log\|\omega\|_p. \tag{7}$$

If

$$t_0 = \int_2^\infty \frac{\bar{\varepsilon}(p)}{p}\,dp, \qquad M_0 = \int_2^\infty \frac{\Gamma(p)}{p}\,dp \tag{8}$$

are both finite, then $e^{-t_0\Delta^k}$ maps L^2 into L^∞ and $\|e^{-t_0\Delta^k}\|_{\infty,2} \leqslant e^{M_0}$.

Proof. Letting $p = p(t)$ vary over the variable t, we define

$$p'(t) = \frac{p}{\bar{\varepsilon}(p)}, \qquad p(0) = 2,$$
$$N'(t) = \frac{p'}{p}\Gamma(p) = \frac{\Gamma(p)}{\bar{\varepsilon}(p)}, \qquad N(0) = 0 \tag{9}$$

and consider the function

$$F(t) = e^{-N(t)}\|\vec{P}_t\omega\|_{p(t)}.$$

We want to prove that $F(t)$ is decreasing in t. As $F(t)$ is positive, it would suffice to show that $\log F(t)$ is decreasing. We compute

$$\frac{d}{dt}\big[\log F(t)\big] = \frac{F'(t)}{F(t)} = -N'(t) + \frac{(\|\vec{P}_t\omega\|_{p(t)})'}{\|\vec{P}_t\omega\|_{p(t)}}$$
$$= -N'(t) + \frac{1}{\|\vec{P}_t\omega\|_{p(t)}}\bigg[-\|\vec{P}_t\omega\|_{p(t)}\frac{p'(t)}{p^2(t)}\log\big(\|\vec{P}_t\omega\|_{p(t)}^{p(t)}\big)$$

$$
\begin{aligned}
&\quad + \frac{1}{p(t)} \|\vec{P}_t\omega\|_{p(t)} \frac{\int_M \frac{d}{dt}[|\vec{P}_t\omega|^{p(t)}]\,dv}{\|\vec{P}_t\omega\|_{p(t)}^{p(t)}}\Bigg] \\
&= -N'(t) - \frac{p'(t)}{p(t)} \log \|\vec{P}_t\omega\|_{p(t)} \\
&\quad + \frac{1}{p(t)\|\vec{P}_t\omega\|_{p(t)}^{p(t)}} \int_M |\vec{P}_t\omega|^{p(t)} \left[p'(t) \log |\vec{P}_t\omega| + p(t) \frac{\frac{d}{dt}[|\vec{P}_t\omega|]}{|\vec{P}_t\omega|} \right] dv \\
&= -N'(t) - \frac{p'(t)}{p(t)} \log \|\vec{P}_t\omega\|_{p(t)} + \frac{p'(t)}{p(t)\|\vec{P}_t\omega\|_{p(t)}^{p(t)}} \int_M |\vec{P}_t\omega|^{p(t)} \log |\vec{P}_t\omega|\,dv \\
&\quad + \frac{1}{2\|\vec{P}_t\omega\|^{p(t)}} \int_M |\vec{P}_t\omega|^{p(t)-2} \frac{d}{dt}\big(|\vec{P}_t\omega|^2\big)\,dv \\
&\leqslant -N'(t) - \frac{p'(t)}{p(t)} \log \|\vec{P}_t\omega\|_{p(t)} \qquad (*) \\
&\quad + \frac{p'(t)}{p(t)\|\vec{P}_t\omega\|_{p(t)}^{p(t)}} \Bigg[\bar{\varepsilon}\big(p(t)\big) \int_M \big(\Delta\big(|\vec{P}_t\omega|\big), |\vec{P}_t\omega|^{p(t)-2}\vec{P}_t\omega\big)\,dv \\
&\quad + \|\vec{P}_t\omega\|_{p(t)}^{p(t)} \log \|\vec{P}_t\omega\|_{p(t)} + \Gamma\big(p(t)\big) \|\vec{P}_t\omega\|_{p(t)}^{p(t)} \Bigg] \\
&\quad + \frac{1}{2\|\vec{P}_t\omega\|_{p(t)}^{p(t)}} \int_M |\vec{P}_t\omega|^{p(t)-2} \frac{d}{dt}\big(|\vec{P}_t\omega|^2\big)\,dv \\
&= -N'(t) - \frac{1}{\|\vec{P}_t\omega\|_{p(t)}^{p(t)}} \Bigg[\frac{\bar{\varepsilon}(p(t))p'(t)}{p(t)} \int_M \big(\Delta\big(|\vec{P}_t\omega|\big), \big(|\vec{P}_t\omega|^{p(t)-2}\vec{P}_t\omega\big)\big)\,dv \\
&\quad - \int_M |\vec{P}_t\omega|^{p(t)-2} \big(\Delta(\vec{P}_t\omega), \vec{P}_t\omega\big)\,dv \Bigg] + \frac{p'(t)}{p(t)} \Gamma\big(p(t)\big).
\end{aligned}
$$

Inequality $(*)$ follows from applying Eq. (7) to the form $\vec{P}_t\omega$ at the constant $p = p(t)$. The final expression in the computation above is actually zero for $p(t)$ and $N(t)$ as in (9), as the terms inside the brackets cancel out and the first term gets cancelled with the last one. Note that the above computations are justified on our domain, which is well defined given that the heat operator is bounded on all of the L^p for $1 \leqslant p \leqslant \infty$.

Therefore, $\frac{F'(t)}{F(t)} \leqslant 0$. As $F(t)$ is positive, then it must be a decreasing function such that

$$
\begin{aligned}
F(t) = e^{-N(t)} \|\vec{P}_t\omega\|_{p(t)} &\leqslant F(0) = e^{-N(0)} \|\vec{P}_0\omega\|_{p(0)} \\
&= \|\omega\|_2 \Rightarrow \|\vec{P}_t\omega\|_{p(t)} \leqslant e^{N(t)} \|\omega\|_2. \qquad (11)
\end{aligned}
$$

For $0 \leqslant t < t_0$, $p(t)$ is monotone increasing. Given that $t_0 = \int_2^\infty \frac{\bar{\varepsilon}(p)}{p}\,dp = \int_2^\infty \frac{dt}{dp}\,dp$ and $p(0) = 2$, we conclude that $p(t) \to \infty$ as $t \to t_0$.

At the same time, $N(t) = \int_0^t \frac{\Gamma(p(t))}{\bar{\varepsilon}(p(t))}\,dt = \int_2^{p(t)} \frac{\Gamma(p)}{p}\,dp$.

Therefore, as $t \to t_0$, $N(t) \to M_0$.

For $0 \leqslant t < t_0$, $F(t)$ is decreasing with respect to t. Thus, for any compact subset B and $s > t$ such that $p(s) > p(t)$ and $\frac{1}{q} + \frac{1}{p(s)} = \frac{1}{p(t)}$

$$\begin{aligned}\|\chi_B \vec{P}_s\omega\|_{p(t)} &\leqslant \big[Vol(B)\big]^{1/q}\|\vec{P}_s\omega\|_{p(s)} \leqslant \big[Vol(B)\big]^{1/q}\|\vec{P}_t\omega\|_{p(t)}e^{N(s)-N(t)}\\ &\leqslant \big[Vol(B)\big]^{1/q}e^{N(s)}\|\omega\|_2\end{aligned}$$

by Eq. (11).

Sending $s, t \to t_0$ we get

$$\|\chi_B \vec{P}_s\omega\|_{p(t)} \to \|\chi_B \vec{P}_{t_0}\omega\|_\infty, \qquad \frac{1}{q} \to 0, \qquad N(s) \to M_0.$$

Therefore, $\|\vec{P}_{t_0}\omega\|_\infty \leqslant e^{M_0}\|\omega\|_2$ for all ω in the domain.

Note that explicit solutions for p and N do exist in the special case where $p(0) = 2$ and for coefficients $\bar{\varepsilon}(p) = \varepsilon\frac{p}{2(p-1)}$ and $\Gamma(p) = \beta'\frac{2}{p} + \sup\{\varepsilon\frac{pK_1}{2(p-1)}, 0\}$ as in Lemma 3.2. They are

$$\begin{aligned}p(t) &= 1 + \exp\left\{\frac{2}{\varepsilon}t\right\},\\ N(t) &= 2\beta'\left(\frac{1}{2} - \frac{1}{p(t)}\right) + \frac{\varepsilon}{2}\sup\{K_1, 0\}\log\big(p(t) - 1\big). \qquad \square\end{aligned}$$

The following step is necessary for the results of Section 4. The aim is to prove a similar bound for an appropriate perturbation of the heat operator of Δ^k. We follow the procedure outlined in Davies [4], while at the same time generalize the results to include the case of k-forms.

To this purpose, we set $\phi = e^{\alpha\psi}$ with $|d\psi|^2 \leqslant 1$ and α real. What we would like to show, is that (Sp) holds for the perturbed operator $T = \phi^{-1} \circ \Delta^k \circ \phi$ on the space of forms $\{\phi^{-1}\omega \mid \omega \in \bigcup_{t>0} e^{-t\Delta^k}(L^1 \cap L^\infty)\}$. The next lemma is crucial in our estimates.

Lemma 3.3. *For any C_c^∞ form ω in M, $2 < p < \infty$, $0 < \mu < 1$, the following inequality holds*

$$\begin{aligned}&(1-\mu)\big\langle \bar{D}\omega, \bar{D}\big(|\omega|^{p-2}\omega\big)\big\rangle\\ &\quad\leqslant \big\langle \bar{D}(\phi\omega), \bar{D}\big(\phi^{-1}|\omega|^{p-2}\omega\big)\big\rangle + \alpha^2\left[1 + \frac{(p-2)^2}{4(p-1)\mu}\right]\|\omega\|_p^p.\end{aligned}$$

Proof. Note that $d\phi = \alpha e^{\alpha\psi}d\psi$ and that $|d\phi|^2 = \alpha^2\phi^2|d\psi|^2 \leqslant \alpha^2\phi^2$. Then:

$$\begin{aligned}&\big\langle \bar{D}(\phi\omega), \bar{D}\big(\phi^{-1}|\omega|^{p-2}\omega\big)\big\rangle\\ &\quad= -\frac{|d\phi|^2}{\phi^2}\|\omega\|_p^p + \big\langle \bar{D}\omega, \bar{D}\big(|\omega|^{p-2}\omega\big)\big\rangle\\ &\qquad - \frac{1}{\phi}\big\langle \bar{D}\omega, (\bar{D}\phi)|\omega|^{p-2}\omega\big\rangle + \left\langle \frac{\bar{D}\phi}{\phi}\omega, \bar{D}\big(|\omega|^{p-2}\omega\big)\right\rangle\end{aligned}$$

$$= -\alpha^2 \|\omega\|_p^p + \langle \bar{D}\omega, \bar{D}(|\omega|^{p-2}\omega)\rangle + \int_M \sum_i \frac{D_{V_i}\phi}{\phi} |\omega|^2 D_{V_i}(|\omega|^{p-2})\, dv$$

$$\geqslant -\alpha^2 \|\omega\|_p^p + \langle \bar{D}\omega, \bar{D}(|\omega|^{p-2}\omega)\rangle - |\alpha| \int_M |\omega|^2 \big|d(|\omega|^{p-2})\big|\, dv. \tag{12}$$

We will try to find an appropriate lower bound for the last term in Eq. (12). For any positive constant δ, Young's Inequality gives

$$\int_M |\omega|^2 \big|d(|\omega|^{p-2})\big|\, dv = (p-2) \int_M |\omega|^{p-1} \big|d|\omega|\big|\, dv$$

$$\leqslant 2(p-2)\delta \|\omega\|_p^p + \frac{(p-2)}{2\delta} \int_M |\omega|^{p-2} \big|d|\omega|\big|^2\, dv. \tag{13}$$

Note that

$$\int_M \big|d|\omega|^{\frac{p}{2}}\big|^2\, dv = \frac{p^2}{4} \int_M |\omega|^{p-2} \big|d|\omega|\big|^2\, dv \tag{a}$$

and using the calculation in Eq. (5)

$$\int_M \big|d|\omega|^{\frac{p}{2}}\big|^2\, dv = \frac{p^2}{(p-2)^2} \Big[\int_M \big|\bar{D}(|\omega|^{\frac{p}{2}-1}\omega)\big|^2 - \langle \omega, \mathcal{L}(|\omega|^{p-2}\omega)\rangle \Big]$$

$$\leqslant \frac{p^2}{4(p-1)} \langle \bar{D}\omega, \bar{D}(|\omega|^{p-2}\omega)\rangle \quad \text{by (4).} \tag{b}$$

By combining the inequalities in (a) and (b),

$$\int_M |\omega|^{p-2} \big|d|\omega|\big|^2\, dv \leqslant \frac{1}{p-1} \langle \bar{D}\omega, \bar{D}(|\omega|^{p-2}\omega)\rangle. \tag{13}$$

We now substitute the estimate from (13) into the right-hand side of (13). This gives a new lower bound for the last term in (12) and the following lower bound

$$\langle \bar{D}(\phi\omega), \bar{D}(\phi^{-1}|\omega|^{p-2}\omega)\rangle$$
$$\geqslant -\Big[\alpha^2 + |\alpha|(p-2)\frac{\delta}{2}\Big] \|\omega\|_p^p + \Big[1 - |\alpha| \frac{p-2}{2\delta} \frac{1}{p-1}\Big] \langle \bar{D}\omega, \bar{D}(|\omega|^{p-2}\omega)\rangle.$$

Choosing δ such that $|\alpha| \frac{p-2}{2\delta} \frac{1}{p-1} = \mu < 1$, we obtain the inequality of the lemma with the right constants, for all $0 < \mu < 1$. $\square$

Corollary 3.1. *If* (S2) *holds for the Hodge Laplacian with coefficients ε and β', then for any C_c^∞ form ω and for all* $0<\mu<1$, $2<p<\infty$

$$\int_M |\omega|^p \log|\omega| \leqslant \tilde{\varepsilon}\big[\big\langle \bar{D}(\phi\omega), \bar{D}\big(\phi^{-1}|\omega|^{p-2}\omega\big)\big\rangle + \big\langle \mathcal{W}(\phi\omega), \phi^{-1}|\omega|^{p-2}\omega\big\rangle\big] + \gamma(\tilde{\varepsilon}, p)\|\omega\|_p^p + \|\omega\|_p^p \log\|\omega\|_p \tag{14}$$

where $\tilde{\varepsilon} = \varepsilon\frac{p}{2(p-1)(1-\mu)}$ *and* $\gamma(\tilde{\varepsilon}, p) = \beta'\frac{2}{p} + \sup\{\tilde{\varepsilon}K_1, 0\} + \tilde{\varepsilon}|\alpha|^2[1+\frac{(p-2)^2}{4(p-1)\mu}]$.

Proof. As demonstrated in Lemma 3.2, if (S2) holds then (Sp) also holds with coefficients as given in the lemma. By combining (Sp) with the inequality of Lemma 3.3 we obtain

$$\int_M |\omega|^p \log|\omega| \leqslant \tilde{\varepsilon}\big\langle \bar{D}(\phi\omega), \bar{D}\big(\phi^{-1}|\omega|^{p-2}\omega\big)\big\rangle + \tilde{\varepsilon}(1-\mu)\big\langle \mathcal{W}(\phi\omega), \phi^{-1}|\omega|^{p-2}\omega\big\rangle + \big[\gamma(\tilde{\varepsilon}, p) - \mu\sup\{\tilde{\varepsilon}K_1, 0\}\big]\|\omega\|_p^p + \|\omega\|_p^p \log\|\omega\|_p.$$

After using the lower bound on $\mathcal{W}$, inequality (14) follows. □

We would like to generalize the results of the corollary above to a larger class of forms, namely to those in: $\mathcal{D} = \phi^{-1}\mathcal{D}_1$, where $\mathcal{D}_1 = \bigcup_{t>0} e^{-t\Delta^k}(L^1 \cap L^\infty)$.

Let us first make some remarks about the space $\mathcal{D}$.

We have defined $T = \phi^{-1} \circ \Delta^k \circ \phi$. As ϕ and ϕ^{-1} are bounded functions, it follows from the remark in the proof of Lemma 3.2 that $\mathcal{D}$ is dense in L^p for all $1 \leqslant p < \infty$.

This implies that $\mathrm{Dom}(T) = \{\omega\colon\ \phi\omega \in \mathrm{Dom}(\Delta^k)\}$.

Furthermore, $\mathcal{D}$ remains invariant under the semigroup e^{-tT} (for details we refer to [1]).

Lemma 3.4. *The inequality of Corollary* 3.1 *can be generalized to hold for all forms in the space $\mathcal{D}$.*

Proof. We can equivalently show that

$$\int_M \big|\phi^{-1}\omega\big|^p \log\big|\phi^{-1}\omega\big| \leqslant \tilde{\varepsilon}\big[\big\langle \bar{D}(\omega), \bar{D}\big(\phi^{-p}|\omega|^{p-2}\omega\big)\big\rangle + \big\langle \mathcal{W}\omega, \phi^{-p}|\omega|^{p-2}\omega\big\rangle\big] + \gamma(\tilde{\varepsilon}, p)\big\|\phi^{-1}\omega\big\|_p^p + \big\|\phi^{-1}\omega\big\|_p^p \log\big\|\phi^{-1}\omega\big\|_p \tag{15}$$

for any $\omega \in \mathcal{D}_1$.

We observe that $\mathcal{D}_1 = \bigcup_{t>0} e^{-t\Delta^k}(L^1 \cap L^\infty) \subseteq \mathrm{Dom}(Q_{\Delta^k}) \cap L^1 \cap L^\infty = \mathcal{D}_2$, so it suffices to prove the above statement for ω in $\mathcal{D}_2$.

On $\mathcal{D}_2$ the left-hand side of inequality (15) is finite, as $\phi^{-1}\omega \in L^1 \cap L^\infty$. A straightforward computation shows

$$Q_{\Delta^k}\big(|\omega|^{p-2}\omega\big) \leqslant (p-1)^2\|\omega\|_\infty^{2p-4} Q_{\Delta^k}(\omega).$$

With this result in mind and the Cauchy–Schwarz inequality, the right-hand side in inequality (15) will be bounded if we are able to show that $\mathcal{D}_2$ remains invariant under multiplication by ϕ^{-p}. In fact, it is enough to show that $\mathrm{Dom}(Q_\Delta)$ remains invariant under multiplication by ϕ^{-p}, since ϕ^{-p} is a bounded function. To demonstrate this, we define the inner product $\langle\omega,\eta\rangle' = \langle\omega,\eta\rangle + \langle\bar{D}\omega,\bar{D}\eta\rangle$ on $\mathrm{Dom}(Q_{\Delta^k})$, and the associated complete norm $\|\omega\|' = \{\langle\omega,\omega\rangle'\}^{1/2}$. A computation shows that on the dense subspace $C_c^\infty(\Lambda^k)$ of $\mathrm{Dom}(Q_{\Delta^k})$, $\|\omega\|'$ and $\|\phi^{-p}\omega\|'$ are equivalent norms, as ϕ and its gradient are bounded. And this completes the proof of the lemma (for further details we also refer the reader to [4]). $\square$

Given the result of Corollary 3.1 for k-forms in $\mathcal{D}$, we may conclude that T is ultracontractive, whenever the constants in the logarithmic Sobolev inequality are appropriate.

Corollary 3.2. *Let $\tilde{\varepsilon}(p) > 0$, $2 < p < \infty$, $\Gamma(p) = \gamma(\tilde{\varepsilon}(p), p)$ be the coefficients of the inequality in Corollary* 3.1. *If*

$$t_0 = \int_2^\infty \frac{\tilde{\varepsilon}(p)}{p}\,dp, \qquad M_0 = \int_2^\infty \frac{\Gamma(p)}{p}\,dp$$

are both finite, then $\exp(-\phi^{-1}\circ\Delta^k\circ\phi t_0)$ *maps L^2 to L^∞ and*

$$\left\|\exp\left(-\phi^{-1}\circ\Delta^k\circ\phi t_0\right)\right\|_{\infty,2} \leqslant e^{M_0}. \tag{16}$$

Proof. The proof is identical to that of Theorem 3.1, but here we consider the operator $T = \phi^{-1}\circ\Delta^k\circ\phi$ with domain $\mathcal{D}$. Inequalities (7) and (14) are parallel, and observe that the calculation in the proof of Theorem 3.1 is not affected by the non-self-adjointness of T.
We then have the estimate (16) for all $\omega\in\mathcal{D}$. By the density of $\mathcal{D}$ in L^p for all $1 \leqslant p < \infty$, we get the desired bound on L^2. $\square$

4. The ultracontractivity of $\vec{P}_t$ and a Gaussian bound for it

The purpose of this section is to show sufficient curvature conditions under which $\vec{P}_t$ is ultracontractive and has a Gaussian upper bound.

Lemma 4.1 will be a key step in this process. The assumption $\mathcal{W}^k \geqslant -K_1$ for the Weitzenböck tensor implies a logarithmic Sobolev inequality for Δ^k whenever we have one for Δ^0. As we have seen in Theorem 3.1, the coefficients of the logarithmic Sobolev inequalities are crucial towards proving ultracontractivity. Lemma 4.1 demonstrates that the coefficients for the (S2) of Δ^k are very similar to the ones of Δ^0. As a result, the ultracontractive bound for $\vec{P}_t$ is almost identical to the one for P_t.

Under the curvature conditions that we impose, a Gaussian bound for the heat kernel of Δ^0 implies that P_t is ultracontractive. By Theorem 4.1 we then get a family of inequalities (S2) for Δ^0. This in turn implies a family of inequalities (S2) for Δ^k and coefficients that prove the ultracontractivity of $\vec{P}_t$, and eventually give the desired Gaussian type bound for its integral kernel (see Theorems 4.2 and 4.3).

We note that the result of Theorem 4.1 holds for Δ^k as well. Together with Theorem 4.2 they demonstrate the equivalence between the ultracontractivity of $\vec{P}_t$ and the existence of a family of inequalities (S2) for Δ^k. Observe however, that Theorem 4.1 holds under less strict curvature assumptions than Theorem 4.2.

We begin by showing how the existence of a logarithmic Sobolev inequality (S2) for Δ^0 implies one for Δ^k.

Lemma 4.1. *Assume that $\mathcal{W}^k \geqslant -K_1$. If the logarithmic Sobolev inequality* (S2) *holds for Δ^0 with coefficients ε, β then* (S2) *also holds for Δ^k with corresponding coefficients ε and $\beta + K_1$. Observe furthermore, that a logarithmic Sobolev inequality also exists for the Bochner Laplacian. The converse however, need not be true.*

Proof. Let η be a k-form on M, and $\{V_i\}_i$ be normal coordinates at a point x_0 with a dual coframe $\{\omega^j\}_j$. We will bound the gradient of the norm of the form pointwise, using the following computation

$$\begin{aligned}\bigl(d|\eta|, d|\eta|\bigr) &= \left(\sum_i \omega^i D_{V_i}|\eta|, \sum_j \omega^j D_{V_j}|\eta|\right) = \bigl(\bar{D}|\eta|, \bar{D}|\eta|\bigr)\\ &= \sum_i \left|\frac{(D_{V_i}\eta, \eta)}{|\eta|}\right|^2 = \sum_i \left(D_{V_i}\eta, \frac{\eta}{|\eta|}\right)^2 \leqslant (\bar{D}\eta, \bar{D}\eta).\end{aligned}$$

However,

$$(\bar{D}\eta, \bar{D}\eta) \leqslant (\bar{D}\eta, \bar{D}\eta) + (\mathcal{W}\eta, \eta) + K_1(\eta, \eta). \tag{17}$$

By integrating we obtain the inequality

$$Q_{\Delta^0}(\eta, \eta) \leqslant Q_{\Delta^k}(\eta, \eta) + K_1\|\eta\|_2^2. \tag{18}$$

The result follows.
Finally note that these inequalities imply that whenever $\eta \in \mathrm{Dom}(Q_{\Delta^k}) \cap L^1 \cap L^\infty$ then $|\eta| \in \mathrm{Dom}(Q_{\Delta^0}) \cap L^1 \cap L^\infty$ as well. □

We will continue by demonstrating a method for obtaining a family of logarithmic Sobolev inequalities for a self-adjoint operator, on either functions or differential forms, whenever its heat operator is ultracontractive.

Theorem 4.1. *Let H be a self-adjoint non-negative operator on a manifold M acting on $L^2(\Lambda^k)$. Suppose that its heat operator e^{-Ht} is bounded from L^∞ to L^∞ and it is ultracontractive with $\|e^{-tH}\|_{\infty,2} \leqslant Ce^{l(t)}$ for all $t > 0$, and with $l(t)$ a continuous function. Then for all $\omega \in \mathrm{Dom}(Q_H) \cap L^1 \cap L^\infty$, $|\omega|^2 \log|\omega|$ is in L^1, and the logarithmic Sobolev inequality*

$$\int_M |\omega|^2 \log|\omega|\, dx \leqslant \varepsilon Q_H(\omega) + l(\varepsilon)\|\omega\|_2^2 + \|\omega\|_2^2 \log\|\omega\|_2 \tag{19}$$

is true for all $\varepsilon > 0$.

Proof. This result is the converse of Lemma 3.2. For the case of functions, the proof is found in Davies (Lemma 2.2.3) where he also makes stronger assumptions on the $l(t)$ [4]. We allow ω to be a differential form and we assume that $\|\omega\|_2 = 1$. We set $\omega_s = e^{-sH}\omega$ and $p(s) = \frac{2t}{t-s}$. $p(s)$ is a continuous differentiable function from $[0,t)$ to $[2,\infty)$ with $p'(0) = \frac{2}{t}$.
We fix the point t, and define the operator $A_z = e^{-tHz}$ from L^2 to $L^2 + L^\infty$ on the set $U = \{z \in \mathbb{C}\colon\ 0 \leqslant \mathrm{Re}(z) \leqslant 1\}$. For x real between 0 and 1, e^{tHx} is bounded on L^2 and the bound is uniform in x. Also e^{tHiy} is a contraction on L^2 for all real y. As a result, for $\eta \in L^2$ and $\varphi \in L^1 \cap L^2$, the operator $\langle A_z\eta, \varphi\rangle$ is uniformly bounded and continuous on U, and analytic in the interior of U.
Then, $\|A_{iy}\eta\|_2 \leqslant M_0\|\eta\|_2$ for all $\eta \in L^2$ and $y \in \mathbb{R}$ with $M_0 = 1$, and $\|A_{1+iy}\eta\|_\infty \leqslant M_1\|\eta\|_2$ for all $\eta \in L^2$ and $y \in \mathbb{R}$, by the ultracontractivity property of e^{-Ht} with $M_1 = e^{l(t)}$.
The Stein Interpolation Theorem implies that:

$$\|A_\tau \eta\|_q \leqslant M_1^\tau M_0^{1-\tau}\|\eta\|_p$$

with $p = 2$ and $\frac{1}{q} = \frac{1-\tau}{2}$. By setting $\tau = \frac{s}{t}$ for $0 \leqslant s < t$

$$\left\|e^{-sH}\omega\right\|_{p(s)}^{p(s)} \leqslant \exp\left\{l(t)p(s)\frac{s}{t}\right\}\|\omega\|_2 \quad \Rightarrow \quad \|\omega_s\|_{p(s)}^{p(s)} \leqslant \exp\left\{l(t)p(s)\frac{s}{t}\right\}$$

where $p(s) = \frac{2t}{t-s} = q$ as above.
As both sides of the above inequality are 1 when $s = 0$, we deduce that

$$\frac{d}{ds}2\left[\|\omega_s\|_{p(s)}^{p(s)}\right]\Big|_{s=0} \leqslant \exp\left\{l(t)p(s)\frac{s}{t}\right\}\left[\frac{l(t)}{t}\big(p(s) + sp'(s)\big)\right]\Bigg|_{s=0} = l(t)\frac{2}{t}. \tag{20}$$

On the other hand,

$$\frac{d}{ds}\left[\|\omega_s\|_{p(s)}^{p(s)}\right] = \int |\omega_s|^{p(s)}\left[p'(s)\log|\omega_s| + p(s)\frac{|\omega_s|'}{|\omega_s|}\right].$$

Combining with (20) at $s = 0$

$$\int |\omega|^2\left[\frac{2}{t}\log|\omega| + 2\frac{(-H\omega,\omega)}{|\omega|^2}\right] \leqslant l(t)\frac{2}{t}$$
$$\Rightarrow \quad \int |\omega|^2\log|\omega| + t\langle -H\omega,\omega\rangle \leqslant l(t).$$

Now substituting ε for t and letting $\omega = \frac{\eta}{\|\eta\|_2}$, we get inequality (19). The fact that the left-hand side of (19) is bounded and that the result is true on the claimed domain, is a straightforward argument and the reader is referred to the proof of Lemma 2.2.3 in Davies [4] for the details. □

We now move to the final part of the section where we will find a specific Gaussian type upper bound for $\vec{P}_t$ and show that it is ultracontractive when a family of logarithmic Sobolev inequalities for Δ^k exists. We will first need an appropriate ultracontractive bound for the heat operator of Δ^0, and then use the process outlined in the beginning of the section. Such a result requires some stricter conditions on the geometry of the manifold. We assume that the Ricci curvature is bounded below by a constant $\mathrm{Ric} \geqslant -K_2$ and that the volume of the balls of radius one has uniform lower bound, $Vol(B_1(x)) \geqslant K_3$ where K_3 is independent of x.

The upper bound for the heat kernel of Δ^0 that we apply is due to Saloff-Coste [10] and it holds under the condition of $\mathrm{Ric} \geqslant -K_2$. The original estimate holds over more generalized operators in divergence form and it is an improvement of similar estimates that were previously published by Cheng, Li and Yau [3]. We state it in our specific case

Lemma 4.2. [10] *On a C^∞ smooth manifold M^N with $\mathrm{Ric} \geqslant -K_2$, for all $t > 0$,*

$$P(t, x_1, x_2) \leqslant V_1^{-1/2} V_2^{-1/2} \exp\left(C\sqrt{K_2 t} - \frac{\rho^2}{4C' t}\right)$$

where C, C' are positive and depend only on N, $V_i = Vol(B_{\sqrt{t}}(x_i))$, and $\rho = d(x_1, x_2)$.

From Bishop's inequality, $V_i^{-1} \leqslant C(N, K_2) Vol^{-1}(B_1(x_i)) \sup\{t^{-N/2}, 1\}$.
We observe that for some constant $C(N, K_2, K_3)$ we get upper estimate:

$$P_t(x, y) \leqslant C(N, K_2, K_3) \sup\{t^{-N/2}, 1\} \exp\left(C\sqrt{K_2 t} - \frac{\rho^2(x, y)}{4C' t}\right).$$

For a function f in L^1 we have

$$\begin{aligned}\left\|e^{-t\Delta^0} f\right\|_\infty &= \left\|\int P_t(x, y) f(y)\, dy\right\|_\infty \leqslant \sup_{x,y}\left|P_t(x, y)\right| \|f\|_1 \\ &\leqslant C(N, K_2, K_3) e^{C(N,K_2)t} \sup\{t^{-N/2}, 1\} \|f\|_1. \end{aligned} \tag{21}$$

In fact, the condition on the volume of balls of radius one is necessary for obtaining a uniform in x, y upper bound for $\sup_{x,y} |P_t(x, y)|$. The lower bound for $P_t(x, y)$ given by Saloff-Coste demonstrates that such a uniform upper bound on the heat kernel immediately implies a uniform lower bound on the volume of balls of radius one [10].

Lemma 4.3. *On smooth manifolds M^N with $\mathrm{Ric} \geqslant -K_2$ and a uniform lower bound K_3 on the volume of balls of radius one, the Laplacian on functions is ultracontractive with the following upper bound on its norm*

$$\left\|e^{-t\Delta^0}\right\|_{\infty,2} \leqslant C(N, K_2, K_3) e^{C(N,K_2)t} \sup\{t^{-N/4}, 1\} \tag{22}$$

for all $t > 0$.

Proof. By the Riesz–Thorin Interpolation Theorem

$$\begin{gathered}\left\|e^{-t\Delta^0} f\right\|_q \leqslant M_1^\tau M_0^{1-\tau} \|f\|_p \quad \text{where} \\ \frac{1}{p} = \frac{\tau}{p_1} + \frac{1-\tau}{p_0}, \qquad \frac{1}{q} = \frac{\tau}{q_1} + \frac{1-\tau}{q_0}, \\ \left\|e^{-t\Delta^0} f\right\|_{q_1,p_1} = M_1, \qquad \left\|e^{-t\Delta^0} f\right\|_{q_0,p_0} = M_0.\end{gathered}$$

We use the fact that $e^{-t\Delta^0}$ is a contraction on L^∞ and the bound given in (21), and set $q_0 = q_1 = p_0 = \infty$, $p_1 = 1$ and $\tau = 1/2$. The ultracontractive bound follows. $\square$

Observe that we can rewrite the ultracontractive bound in (22) as $\exp S(t)$, where

$$S(t) = C(N, K_2, K_3) + C(N, K_2)t + \sup\left\{-\frac{N}{4}\log t, 0\right\}. \tag{23}$$

Furthermore, this $S(t)$ is a decreasing function for small t. As a result, Theorem 4.1 applies and we obtain a family of logarithmic Sobolev inequalities (S2) for Δ^0 with coefficients ε and $\beta(\varepsilon) = S(\varepsilon)$.

We will now describe the family of logarithmic Sobolev inequalities for the Hodge Laplacian.

Corollary 4.1. *Let M^N be a smooth manifold with* Ric $\geqslant -K_2$ *and $Vol(B_1(x)) \geqslant K_3$ where K_3 is independent of x. When $\mathcal{W}^k \geqslant -K_1$ we can apply Theorem* 4.1 *to the Laplacian on functions, and via Lemma* 4.1 *we get a family of logarithmic Sobolev inequalities* (S2) *for Δ^k, with constant $\beta(\varepsilon) = S(\varepsilon) + \varepsilon K_1$ for all $\varepsilon > 0$.*
Therefore, (Sp) *also holds for Δ^k with $a = \bar{\varepsilon}(p) = \varepsilon\frac{p}{2(p-1)}$ and $b = \Gamma(p) = (S(\varepsilon) + \varepsilon K_1)\frac{2}{p} + \varepsilon\frac{p}{2(p-1)}\sup\{K_1, 0\}$, for all k-forms in $\mathcal{D}_1 = \bigcup_{t>0} e^{-t\Delta^k}(L^1 \cap L^\infty)$, by Lemma* 3.2.

The constant M_0 that appears in Theorem 3.1 can now be calculated using the constant $S(\varepsilon)$ we obtained above.

$$M_0(\varepsilon) = \int_2^\infty \frac{b}{p}\,dp \leqslant \int_2^\infty \left[\frac{2}{p^2}\left(C_{2,3} + \varepsilon C_2 + \sup\left\{-\frac{N}{4}\log\varepsilon, 0\right\} + \varepsilon K_1\right) + \bar{\varepsilon}\frac{K_1}{p}\right]dp.$$

As the equation holds for any positive ε, we can set $\bar{\varepsilon}(p) = \frac{2t}{p}$ for any $t > 0$ and $p \in [2, \infty)$. For $p \geqslant 2$, $\frac{1}{2} \leqslant \frac{p}{2(p-1)} \leqslant 1$, thus $\log\varepsilon = \log\bar{\varepsilon} + \log[\frac{2(p-1)}{p}] \geqslant \log\bar{\varepsilon}$. Moreover, $\bar{\varepsilon} \leqslant \varepsilon \leqslant 2\bar{\varepsilon}$. With the above estimates in mind we get the following upper bound for M_0

$$\begin{aligned} M_0(t) &\leqslant \int_0^t \left[\frac{1}{t}\left[C_{2,3} + 2\bar{\varepsilon}C_2 + \sup\left\{-\frac{N}{4}\log\bar{\varepsilon}, 0\right\}\right] + 2K_1\right]d\bar{\varepsilon} \\ &\leqslant C(N, K_2, K_3) + C(N, K_1, K_2)t + \frac{N}{4}\sup\{-\log t, 0\}. \end{aligned}$$

Furthermore, we compute t_0 to be $t_0 = \int_2^\infty \frac{\bar{\varepsilon}(p)}{p}\,dp = \int_0^t d\bar{\varepsilon} = t$.

The results of Lemma 3.2 and the above computations demonstrate the following theorem

Theorem 4.2. *On smooth manifolds with $\mathcal{W}^k \geqslant -K_1$,* Ric $\geqslant -K_2$ *and $Vol(B_1(\cdot)) \geqslant K_3$, we may obtain via the method of logarithmic Sobolev inequalities an ultracontractive bound for $\vec{P}_t$ of the following type*

$$\left\|\exp\left\{-t\Delta^k\right\}\right\|_{\infty,2} \leqslant \exp\left\{M(t)\right\}$$

where $M(t) = C(N, K_2, K_3) + C(N, K_1, K_2)t + \sup\{-\frac{N}{4}\log t, 0\}$.

After carrying out the above estimates for the case of the perturbed Hodge Laplacian $T = \phi^{-1} \circ \Delta \circ \phi$, the Gaussian upper bound for the heat kernel on forms will follow. As the logarithmic Sobolev inequality (S2) holds for manifolds satisfying the conditions of Corollary 4.1 with $\beta(\varepsilon) = S(\varepsilon) + \varepsilon K_1$, Lemma 3.4 implies that inequality (14) also holds for these manifolds on $\mathcal{D}$ with constants $\tilde{\varepsilon}$ and $\Gamma(p) = \gamma(\tilde{\varepsilon}, p)$ as in Corollary 3.1 and with $\beta'(\varepsilon) = S(\varepsilon) + \varepsilon K_1$.

Applying Corollary 3.2, we will prove that $\exp\{\phi^{-1} \circ \Delta \circ \phi t\}$ is ultracontractive by showing that t_0 and M_0 are finite.

Recall that $t_0 = \int_2^\infty \frac{\tilde{\varepsilon}(p)}{p}\, dp$, $M_0 = \int_2^\infty \frac{\Gamma(p)}{p}\, dp$.

For some $\lambda > 1$ we set $\tilde{\varepsilon}(p) = \lambda 2^\lambda t p^{-\lambda}$ for any $t > 0$ and $p \in [2, \infty)$.

We also fix $\mu = \frac{1}{2}$, so $\tilde{\varepsilon} = \frac{p}{p-1}\varepsilon \leqslant 2\varepsilon$ for $p \geqslant 2$. As a result, $\log \varepsilon + \log 2 \geqslant \log \tilde{\varepsilon}$, and $\varepsilon = \frac{p-1}{p}\tilde{\varepsilon} \leqslant \tilde{\varepsilon}$. Moreover, for $p \geqslant 2$, $\{1 + \frac{(p-2)^2}{2(p-1)}\} \leqslant \frac{p}{2}$.

With these estimates in mind, $t_0 = \int_2^\infty \frac{\tilde{\varepsilon}}{p}\, dp = \int_0^{\lambda t} \frac{1}{\lambda}\, d\tilde{\varepsilon} = t$, just as in Corollary 4.1, and we will find an upper bound for M_0

$$\begin{aligned} M_0(t) &= \int_2^\infty \frac{\Gamma(p)}{p}\, dp \\ &\leqslant \int_2^\infty \left[\frac{2}{p^2}\left[C_{2,3} + \tilde{\varepsilon}(C_2 + K_1) + \sup\left\{-\frac{N}{4}\log\tilde{\varepsilon}, 0\right\}\right] + \frac{\tilde{\varepsilon}}{p}K_1 + \frac{\tilde{\varepsilon}}{2}|\alpha|^2\right] dp \\ &\leqslant C_{\lambda,2,3} + \sup\left\{-\frac{N}{4}\log(t), 0\right\} + t\left[C_{1,2} + \frac{\lambda}{\lambda - 1}|\alpha|^2\right]. \end{aligned}$$

We choose λ so that $\frac{\lambda}{\lambda-1} = 1 + \delta$. Now we are ready to state and prove the main theorem of this section.

Theorem 4.3. *Let M^N be a smooth manifold with $\mathcal{W}^k \geqslant -K_1$, $\mathrm{Ric} \geqslant -K_2$ and $Vol(B_1(x)) \geqslant K_3$ at all points x. Then the heat operator of the Hodge Laplacian on differential k-forms has an integral kernel $\vec{P}(t, x, y)$, with a Gaussian type upper bound such that*

$$\begin{aligned} \left|\vec{P}(t, x, y)\right| &\leqslant C_\delta(N, K_2, K_3) \sup\left\{t^{-N/2}, 1\right\} \\ &\quad \times \exp\bigl(C(N, K_1, K_2)t\bigr) \exp\left[-\frac{d(x, y)^2}{4(1+\delta)t}\right] \end{aligned} \tag{24}$$

for all $0 < t < \infty$, $0 < \delta < 1$, *and* $x, y \in M$.

Proof. The preceding computations together with Corollary 3.2 imply that

$$\begin{aligned} \left\|\exp\left\{-\phi^{-1} \circ \Delta^k \circ \phi t\right\}\right\|_{\infty,2} &\leqslant C_\delta(N, K_2, K_3) \sup\left\{t^{-\frac{N}{4}}, 1\right\} \\ &\quad \times \exp\left\{C(N, K_1, K_2)t + (1+\delta)|\alpha|^2 t\right\}. \end{aligned}$$

Note that $\exp\{-\phi^{-1} \circ \Delta^k \circ \phi t\} = \phi^{-1} \circ e^{-\Delta^k t} \circ \phi$, and recall that $\phi = e^{\alpha\psi}$ with $\alpha \in \mathbb{R}$ and ψ a C^∞ bounded function with $|d\psi|^2 \leqslant 1$. Replacing α by $-\alpha$ and taking adjoints, $\|\phi^{-1} \circ e^{-\Delta^k t} \circ \phi\|_{2,1} = \|\phi^{-1} \circ e^{-\Delta^k t} \circ \phi\|_{\infty,2}$.
By multiplying the two norms together,

$$\left\|\phi^{-1} \circ e^{-2\Delta^k t} \circ \phi\omega\right\|_{\infty,1} \leqslant \left\|\phi^{-1} \circ e^{-\Delta^k t} \circ \phi\right\|_{\infty,2} \left\|\phi^{-1} \circ e^{-\Delta^k t} \circ \phi\right\|_{2,1}$$

and

$$\begin{aligned}\left\|\phi^{-1} \circ e^{-\Delta^k t} \circ \phi\right\|_{\infty,1} \leqslant{}& C_\delta(N, K_2, K_3) \sup\left\{t^{-\frac{N}{2}}, 1\right\} \\ &\times \exp\left\{C(N, K_1, K_2)t + 2(1+\delta)|\alpha|^2 t\right\}.\end{aligned}$$

We conclude that $\phi^{-1} \circ e^{-\Delta^k t} \circ \phi$ has an integral kernel of the form $\phi^{-1}(y) K_{IJ}(t, x, y)\phi(x)$. This kernel is pointwise bounded above, when the basis at x, y is orthonormal, by $C_\delta \sup\{t^{-\frac{N}{2}}, 1\} \times \exp\{C't + 2(1+\delta)|\alpha|^2 t\}$. For a proof that such a kernel exists whenever we have an operator on differential forms that is bounded from L^1 to L^∞ see [2].
As a result,

$$\left|K_{IJ}(t, x, y)\right| \leqslant C_\delta \sup\left\{t^{-\frac{N}{2}}, 1\right\} \exp\left\{C't + 2(1+\delta)|\alpha|^2 t + \alpha\big(\psi(y) - \psi(x)\big)\right\}.$$

If we choose $\alpha = \frac{\psi(x)-\psi(y)}{2\sqrt{2}(1+\delta)t}$ and observe that $|\psi(x) - \psi(y)|^2 \leqslant d(x, y)^2$, we get the desired bound of Theorem 4.3. □

The critical inequalities that allowed us to demonstrate the Gaussian bounds of the Hodge Laplacian were inequality (17), which gave a pointwise comparison of the inner products of the Laplacian on functions and the Hodge Laplacian, and inequality (18) which gave a similar comparison of their quadratic forms. For possible extensions of the results of this article one could consider self-adjoint operators H on smooth bundles over M that satisfy similar pointwise comparison properties of the form

$$\big(\Delta^0|\omega|, |\omega|\big) \leqslant (H\omega, \omega) + K_1|\omega|^2$$

for some constant K_1 and for any L^2 section ω of the bundle. As a result, Lemma 4.1 would hold for this operator. It would also be important for H to have a bounded heat kernel on L^p and to satisfy an equivalent version of Lemma 3.2. In other words, it would be necessary to prove that an (S2) logarithmic Sobolev inequality implies an (Sp) inequality for all $2 < p < \infty$.

We end this paper by comparing our result to that of Rosenberg [8]. It says that on manifolds with $\mathcal{W}^k \geqslant -K_1$ and with Ricci curvature bounded below,

$$\left|\vec{P}(t, x, y)\right| \leqslant e^{tK_1} P(t, x, y).$$

Together with the heat kernel bound due to Saloff-Coste, it gives a more general bound for the heat kernel of Δ^k as follows

$$\left|\vec{P}(t,x,y)\right| \leqslant C(N,K_2)\phi(x)\phi(y)\sup\left\{t^{-N/2},1\right\} \times \exp\left(+K_1 t + 2\sqrt{K_2 t}\right)\exp\left[-\frac{d(x,y)^2}{4C'(N)t}\right]. \tag{25}$$

The intermediate step of using logarithmic Sobolev inequalities forced us to drop the pointwise dependence of the heat kernel bound in the coefficients, in favor of more global constants for the manifold.

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